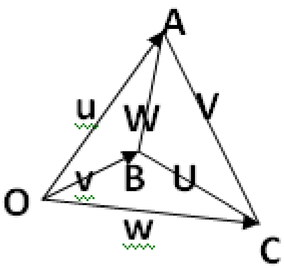
VOLUMEN DEL PARALELEPÍPEDO DETERMINADO POR LOS PUNTOS ABCD

$$(VOL(ABCD))^2 = \frac{1}{8}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & AB^2 & AC^2 & AD^2 \\ 1 & AB^2 & 0 & BC^2 & BD^2 \\ 1 & AC^2 & BC^2 & 0 & CD^2 \\ 1 & AD^2 & BD^2 & CD^2 & 0 \end{vmatrix}$$

Demostración de que el determinante de Cayley nos da el volumen del paralelepípedo:



$$\mathrm{Vol}^2 = \det\begin{pmatrix} u \\ v \\ w \end{pmatrix}\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix}^t = \begin{vmatrix} u^2 & u\cdot v & u\cdot w \\ u\cdot v & v^2 & v\cdot w \\ u\cdot w & v\cdot w & w^2 \end{vmatrix}$$

Y teniendo en cuenta que $W^2 = (u - v)^2 = u^2 + v^2 - 2\, u\cdot v$,

$$u\cdot v = \frac{u^2 + v^2 - W^2}{2}$$

$$\mathrm{Vol}^2 = \begin{vmatrix} u^2 & \frac{u^2+v^2-W^2}{2} & \frac{u^2+w^2-V^2}{2} \\ \frac{u^2+v^2-W^2}{2} & v^2 & \frac{v^2+w^2-U^2}{2} \\ \frac{u^2+w^2-V^2}{2} & \frac{v^2+w^2-U^2}{2} & w^2 \end{vmatrix} =$$

$$\frac{1}{8}\begin{vmatrix} 2u^2 & u^2+v^2-W^2 & u^2+w^2-V^2 \\ u^2+v^2-W^2 & 2v^2 & v^2+w^2-U^2 \\ u^2+w^2-V^2 & v^2+w^2-U^2 & 2w^2 \end{vmatrix} =$$

$$\overset{\text{Añado una col}}{=} \frac{1}{8}\begin{vmatrix} 0 & 2u^2 & u^2+v^2-W^2 & u^2+w^2-V^2 \\ 0 & u^2+v^2-W^2 & 2v^2 & v^2+w^2-U^2 \\ 0 & u^2+w^2-V^2 & v^2+w^2-U^2 & 2w^2 \\ -1 & u^2 & v^2 & w^2 \end{vmatrix} \overset{\text{Resto la fila 4 a las otras}}{=}$$

$$= \frac{1}{8}\begin{vmatrix} 1 & u^2 & u^2-W^2 & u^2-V^2 \\ 1 & v^2-W^2 & v^2 & v^2-U^2 \\ 1 & w^2-V^2 & w^2-U^2 & w^2 \\ -1 & u^2 & v^2 & w^2 \end{vmatrix} \overset{\text{Añado una fila}}{=}$$

$$\frac{1}{8}\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & u^2 & u^2-W^2 & u^2-V^2 & u^2 \\ 1 & v^2-W^2 & v^2 & v^2-U^2 & v^2 \\ 1 & w^2-V^2 & w^2-U^2 & w^2 & w^2 \\ -1 & u^2 & v^2 & w^2 & 0 \end{vmatrix} \overset{\substack{\text{Resto Col 5 a la} \\ \text{2, 3 y 4}}}{=}$$

$$= \frac{1}{8}\begin{vmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -W^2 & -V^2 & u^2 \\ 1 & -W^2 & 0 & -U^2 & v^2 \\ 1 & -V^2 & -U^2 & 0 & w^2 \\ -1 & u^2 & v^2 & w^2 & 0 \end{vmatrix} = \frac{1}{8}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & W^2 & V^2 & u^2 \\ 1 & W^2 & 0 & U^2 & v^2 \\ 1 & V^2 & U^2 & 0 & w^2 \\ 1 & u^2 & v^2 & w^2 & 0 \end{vmatrix} =$$

$$= \boxed{\frac{1}{8}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & AB^2 & AC^2 & AO^2 \\ 1 & AB^2 & 0 & BC^2 & BO^2 \\ 1 & AC^2 & BC^2 & 0 & CO^2 \\ 1 & AO^2 & BO^2 & CO^2 & 0 \end{vmatrix}}$$