Differential Geometry and its Applications 28 (2010) 697–704

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Quintic parametric polynomial minimal surfaces and their properties

Gang Xu [a,c,*], Guo-zhao Wang [b]

[a] Hangzhou Dianzi University, Hangzhou 310018, PR China
[b] Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China
[c] Galaad, INRIA Sophia-Antipolis, 2004 Route des Lucioles, 06902 Sophia-Antipolis Cedex, France

ARTICLE INFO

Article history:
Received 28 November 2009
Received in revised form 14 April 2010
Available online 31 July 2010
Communicated by Z. Shen

MSC:
53A10
49Q05
49Q10

Keywords:
Minimal surface
Harmonic surfaces
Isothermal parametric surface
Isometric minimal surfaces

ABSTRACT

In this paper, quintic parametric polynomial minimal surface and their properties are discussed. We first propose the sufficient condition of quintic harmonic polynomial parametric surface being a minimal surface. Then several new models of minimal surfaces with shape parameters are derived from this condition. We also study the properties of new minimal surfaces, such as symmetry, self-intersection on symmetric planes and containing straight lines. Two one-parameter families of isometric minimal surfaces are also constructed by specifying some proper shape parameters.

1. Introduction

Since Lagrange derived the minimal surface equation in $\boldsymbol{R}^3$ in 1762, minimal surfaces have a long history of over 200 years. A minimal surface is a surface with vanishing mean curvature. As the mean curvature is the variation of area functional, minimal surfaces include the surfaces minimizing the area with a fixed boundary. Because of their attractive properties, minimal surfaces have been extensively employed in many areas such as architecture, material science, aviation, ship manufacture, biology and so on. For instance, the shape of the membrane structure, which has appeared frequently in modern architecture, is mainly based on minimal surfaces [1]. Furthermore, triply periodic minimal surfaces naturally arise in a variety of systems, including nanocomposites, lipid–water systems and certain cell membranes [11].

However, most of the classic minimal surfaces, such as helicoid and catenoid, cannot be represented by Bézier surface or B-spline surface [13]. Because parametric polynomial surface is one of the fundamental elements in CAD systems [5], it is important to find some minimal surfaces with parametric polynomial form. As we know, plane is the unique quadratic minimal surface, Enneper surface is the unique cubic minimal surface [14], and there are few research work on the parametric form of polynomial minimal surface with higher degree. In this paper, we will study the parametric form of quintic parametric polynomial minimal surfaces and their properties.

* Corresponding author at: Hangzhou Dianzi University, Hangzhou 310018, PR China.
E-mail address: xugangzju@gmail.com (G. Xu).

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2010.07.003

1.1. Related work

There have been many literatures on minimal surface in classical differential geometry [15]. The Weierstrass representation is a unified form of minimal surface [14]. However, it is difficult for CAD users to choose proper initial functions for the parametric polynomial minimal surfaces. The discrete minimal surface has been introduced in recent years [2,3,16,19,21]. As the topics which are related with the minimal surface, the computational algorithms for conformal structure on discrete surface are presented in [7–10,12]. There is a close relation between conformal structure and isothermal coordinates. Isothermal coordinates on a Riemannian manifold are local coordinates where the Riemannian metric is locally conformal equivalent to the Euclidean flat metric [7,8]. Some discrete approximation of smooth differential operators are proposed in [23,24]. The Plateau Bézier/B-spline problems are studied in [22] by using Geometric PDE method. A kind of quartic polynomial minimal surface is presented in [13], but their properties haven't been discussed. Applications of minimal surface in aesthetic design, 3D ball skinning, aviation and nanostructures modeling have been presented in [6,17,18,20].

1.2. Contributions and overview

Our main contributions are:

- For quintic case, we propose sufficient conditions of harmonic polynomial parametric surface being minimal surface. The coefficient relations are derived from the isothermal condition.
- Based on this condition, several families of new minimal surfaces with shape parameters are presented. We analyze the geometric properties of the new minimal surfaces, such as symmetry, self-intersection on symmetric planes and containing straight lines.
- From these minimal surfaces, we can construct one-parameter family of isometric minimal surfaces by specifying proper shape parameters.

The remainder of this paper is organized as follows. Some preliminaries and notations are presented in Section 2. Section 3 presents the sufficient conditions for quintic harmonic polynomial surface being minimal surface. Section 4 derives a new family of minimal surface from the condition, and studies its properties. Finally, we conclude and list some future work in Section 5.

2. Preliminary

In this section, we shall review some concepts and results related to minimal surfaces [4,14].

If the parametric form of a regular patch in $\boldsymbol{R}^3$ is given by

$$\boldsymbol{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big), \quad u \in (-\infty,+\infty),\ v \in (-\infty,+\infty).$$

Then the coefficients of first fundamental form of $\boldsymbol{r}(u,v)$ are

$$E = \langle \boldsymbol{r}_u, \boldsymbol{r}_u \rangle, \qquad F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v \rangle, \qquad G = \langle \boldsymbol{r}_v, \boldsymbol{r}_v \rangle,$$

where $\boldsymbol{r}_u$, $\boldsymbol{r}_v$ are the first-order partial derivatives of $\boldsymbol{r}(u,v)$ with respect to u and v respectively and $\langle , \rangle$ defines the dot product of the vectors. The coefficients of second fundamental form of $\boldsymbol{r}(u,v)$ are

$$L = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{uu}), \qquad M = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{uv}), \qquad N = (\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_{vv}),$$

where $\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{vv}$ and $\boldsymbol{r}_{uv}$ are the second-order partial derivatives of $\boldsymbol{r}(u,v)$ and $(,,)$ defines the mixed product of the vectors. Then the mean curvature H and the Gaussian curvature K of $\boldsymbol{r}(u,v)$ are

$$H = \frac{EN - 2FM + LG}{2(EG - F^2)}, \qquad K = \frac{LN - M^2}{EG - F^2}.$$

Definition 1. If $\boldsymbol{r}(u,v)$ satisfies $E = G$, $F = 0$, then $\boldsymbol{r}(u,v)$ is called *surface with isothermal parameterizations*.

Definition 2. If $\boldsymbol{r}(u,v)$ satisfies $\boldsymbol{r}_{uu} + \boldsymbol{r}_{vv} = 0$, then $\boldsymbol{r}(u,v)$ is called *harmonic surface*.

Definition 3. If $\boldsymbol{r}(u,v)$ satisfies $H = 0$, then $\boldsymbol{r}(u,v)$ is called *minimal surface*.

Lemma 1. *The surface with isothermal parameter is minimal surface if and only if it is harmonic surface.*

Definition 4. If two differentiable functions $p(u,v), q(u,v): \boldsymbol{U} \mapsto \boldsymbol{R}$ satisfy the Cauchy–Riemann equations

$$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial v}, \qquad \frac{\partial p}{\partial v} = -\frac{\partial q}{\partial u},$$

and both are harmonic, then the functions are said to be *harmonic conjugate*.

Definition 5. If $\boldsymbol{P} = (p_1, p_2, p_3)$ and $\boldsymbol{Q} = (q_1, q_2, q_3)$ are with isothermal parameterizations such that p_k and q_k are harmonic conjugate for $k = 1, 2, 3$, then $\boldsymbol{P}$ and $\boldsymbol{Q}$ are said to be *parametric conjugate minimal surfaces*.

For example, helicoid and catenoid are a pair of conjugate minimal surface. A pair of conjugate minimal surfaces satisfy the following lemma.

Lemma 2. *Given two conjugate minimal surfaces $\boldsymbol{P}$ and $\boldsymbol{Q}$ and a real number t, all surfaces of the one-parameter family*

$$\boldsymbol{P}_t = (\cos t)\boldsymbol{P} + (\sin t)\boldsymbol{Q}$$

satisfy

(a) *$\boldsymbol{P}_t$ are minimal surfaces for all $t \in \boldsymbol{R}$;*
(b) *$\boldsymbol{P}_t$ have the same first fundamental forms for $t \in \boldsymbol{R}$.*

From Lemma 2, any pair of conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces, and the first fundamental form of this family is independent of t. In other words, these minimal surfaces are isometric and have the same Gaussian curvatures at corresponding points.

3. Sufficient condition for quintic parametric polynomial minimal surface

In this section, we will present a sufficient condition for quintic parametric polynomial minimal surface. The sufficient condition is derived from Lemma 1. Hence, quintic harmonic parametric polynomial surfaces should be studied firstly. In fact, we have the following lemma.

Lemma 3. *Quintic harmonic polynomial surface $\boldsymbol{r}(u,v)$ must have the following form*

$$\begin{aligned}\boldsymbol{r}(u,v) = {} & \boldsymbol{a}\left(u^5 - 10u^3v^2 + 5uv^4\right) + \boldsymbol{b}\left(v^5 - 10u^2v^3 + 5u^4v\right) + \boldsymbol{c}\left(u^4 - 6u^2v^2 + v^4\right) \\ & + \boldsymbol{d}uv\left(u^2 - v^2\right) + \boldsymbol{e}u\left(u^2 - 3v^2\right) + \boldsymbol{f}v\left(v^2 - 3u^2\right) + \boldsymbol{g}\left(u^2 - v^2\right) + \boldsymbol{h}uv + \boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k},\end{aligned}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are coefficient vectors.

Proof. Let $\boldsymbol{r}(u,v) = \sum_{0 \leqslant k+l \leqslant 5}^{5} \boldsymbol{w}_{kl}u^kv^l$. For simplicity, we only consider the terms with $k + l = 5$. From the harmonic condition, we have

$$\boldsymbol{w}_{32} = -10\boldsymbol{w}_{50}, \qquad \boldsymbol{w}_{14} = 5\boldsymbol{w}_{50}, \qquad 10\boldsymbol{w}_{23} = -10\boldsymbol{w}_{05}, \qquad \boldsymbol{w}_{41} = 5\boldsymbol{w}_{05}.$$

Hence, if $\boldsymbol{w}_{50} = \boldsymbol{a}$, $\boldsymbol{w}_{05} = \boldsymbol{b}$, then

$$\boldsymbol{w}_{32} = -10\boldsymbol{a}, \qquad \boldsymbol{w}_{14} = 5\boldsymbol{a}, \qquad \boldsymbol{w}_{23} = -10\boldsymbol{b}, \qquad \boldsymbol{w}_{41} = 5\boldsymbol{b}.$$

Analogously, the relations of coefficients of other terms with $0 \leqslant k + l < 5$ can be obtained. The proof is completed. □

From Lemmas 1 and 3, the sufficient condition can be easily obtained.

Theorem 1. *If the coefficient vectors of quintic harmonic polynomial surface $\boldsymbol{r}(u,v)$ in Lemma 3 satisfy the following system of equations*

$$\begin{cases} \boldsymbol{a}^2 = \boldsymbol{b}^2, \\ \boldsymbol{a}\cdot\boldsymbol{b} = 0, \\ 4\boldsymbol{a}\cdot\boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{d} = 0, \\ \boldsymbol{a}\cdot\boldsymbol{d} + 4\boldsymbol{b}\cdot\boldsymbol{c} = 0, \\ 16\boldsymbol{c}^2 - \boldsymbol{d}^2 + 30\boldsymbol{a}\cdot\boldsymbol{e} + 30\boldsymbol{b}\cdot\boldsymbol{f} = 0, \\ 4\boldsymbol{d}\cdot\boldsymbol{c} + 15\boldsymbol{b}\cdot\boldsymbol{e} - 15\boldsymbol{a}\cdot\boldsymbol{f} = 0, \\ 9\boldsymbol{e}^2 - 9\boldsymbol{f}^2 + 16\boldsymbol{c}\cdot\boldsymbol{g} - 2\boldsymbol{d}\cdot\boldsymbol{h} + 10\boldsymbol{a}\cdot\boldsymbol{i} - 10\boldsymbol{b}\cdot\boldsymbol{j} = 0, \\ 9\boldsymbol{e}\cdot\boldsymbol{f} - 4\boldsymbol{c}\cdot\boldsymbol{h} - 2\boldsymbol{d}\cdot\boldsymbol{g} - 5\boldsymbol{b}\cdot\boldsymbol{i} - 5\boldsymbol{a}\cdot\boldsymbol{j} = 0, \\ 4\boldsymbol{g}^2 - \boldsymbol{h}^2 + 6\boldsymbol{e}\cdot\boldsymbol{i} + 6\boldsymbol{f}\cdot\boldsymbol{j} = 0, \\ 2\boldsymbol{g}\cdot\boldsymbol{h} - 3\boldsymbol{f}\cdot\boldsymbol{i} + 3\boldsymbol{e}\cdot\boldsymbol{j} = 0, \\ 5\boldsymbol{a}\cdot\boldsymbol{h} + 10\boldsymbol{b}\cdot\boldsymbol{g} - 12\boldsymbol{c}\cdot\boldsymbol{f} + 3\boldsymbol{d}\cdot\boldsymbol{e} = 0, \\ 5\boldsymbol{b}\cdot\boldsymbol{h} - 10\boldsymbol{a}\cdot\boldsymbol{g} - 3\boldsymbol{d}\cdot\boldsymbol{f} - 12\boldsymbol{c}\cdot\boldsymbol{e} = 0, \\ 2\boldsymbol{e}\cdot\boldsymbol{g} + \boldsymbol{f}\cdot\boldsymbol{h} = 0, \\ 2\boldsymbol{f}\cdot\boldsymbol{g} - \boldsymbol{e}\cdot\boldsymbol{h} = 0, \\ \boldsymbol{h}\cdot\boldsymbol{i} + 2\boldsymbol{g}\cdot\boldsymbol{j} = 0, \\ 2\boldsymbol{g}\cdot\boldsymbol{i} - \boldsymbol{h}\cdot\boldsymbol{j} = 0, \\ \boldsymbol{i}^2 = \boldsymbol{j}^2, \\ \boldsymbol{i}\cdot\boldsymbol{j} = 0, \end{cases} \tag{1}$$

then $\boldsymbol{r}(u,v)$ is a minimal surface.

Proof. The partial derivatives of the harmonic surface $\boldsymbol{r}(u,v)$ in Lemma 3 have the following forms:

$$\boldsymbol{r}_u(u,v) = 5\boldsymbol{a}A_4^e + 10\boldsymbol{b}A_4^o + 4\boldsymbol{c}A_3^o + \boldsymbol{d}A_3^e + 3\boldsymbol{e}A_2^e - 6\boldsymbol{f}A_2^o + 2\boldsymbol{g}A_1^o + \boldsymbol{h}A_1^e + \boldsymbol{i},$$
$$\boldsymbol{r}_v(u,v) = 5\boldsymbol{b}A_4^e - 10\boldsymbol{a}A_4^o + \boldsymbol{d}A_3^o - 4\boldsymbol{c}A_3^e - 3\boldsymbol{f}A_2^e - 6\boldsymbol{e}A_2^o + \boldsymbol{h}A_1^o - 2\boldsymbol{g}A_1^e + \boldsymbol{j}$$

where $A_4^e = u^4 - 6u^2v^2 + v^4$, $A_4^o = 2u^3v - 2uv^3$, $A_3^o = u^3 - 3uv^2$, $A_3^e = 3u^2v - v^3$, $A_2^e = u^2 - v^2$, $A_2^o = uv$, $A_1^o = u$, $A_1^e = v$.

Hence, from $F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v \rangle$, the term u^8 in F is related with A_4^e, then we obtain $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ from $F = 0$. The term u^7v is related with A_4^e and A_4^o, then we get $\boldsymbol{a}^2 = \boldsymbol{b}^2$. Similarly, the other equations in (1) can be obtain from $F = 0$ and $E = G$.

It is noted that we obtain only two equations for the terms u^iv^j, $i + j = k$, $k = 0, 1, 2, \ldots, 7, 8$. One is for the case of i is even, and the other one is for the case of i is odd. The equations derived from $F = 0$ are the same as the case of $E = G$ except for the equations $\boldsymbol{i}^2 = \boldsymbol{j}^2$ and $\boldsymbol{i}\cdot\boldsymbol{j} = 0$. Hence, we can get 18 equations from the isothermal condition. □

4. Examples and properties

In this section, we will derive some kinds of minimal surface from the sufficient condition and study their properties.

It is difficult to find the general solution for the system (1). But some special solutions can be constructed from the condition. In order to simplify the system (1), we firstly make some assumptions about the coefficient vectors,

$$\boldsymbol{a} = (a_1, a_2, 0), \quad \boldsymbol{b} = (-a_2, a_1, 0), \quad \boldsymbol{c} = (c_1, c_2, c_3), \quad \boldsymbol{d} = (d_1, d_2, d_3), \quad \boldsymbol{e} = (e_1, e_2, e_3),$$
$$\boldsymbol{f} = (f_1, f_2, f_3), \quad \boldsymbol{g} = (g_1, g_2, g_3), \quad \boldsymbol{h} = (h_1, h_2, h_3), \quad \boldsymbol{i} = (i_1, i_2, i_3), \quad \boldsymbol{j} = (j_1, j_2, j_3).$$

Supposing $\boldsymbol{g} = \boldsymbol{h} = \boldsymbol{i} = \boldsymbol{j} = 0$, $c_1 = c_2 = d_1 = d_2 = e_3 = f_3 = 0$, we obtain

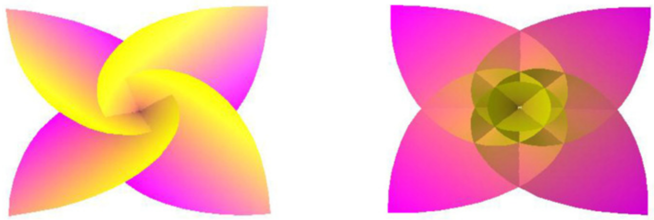
$$\begin{cases} 16c_3^2 - d_3^2 + 30a_1e_1 + 30a_2e_2 - 30a_2f_1 + 30a_1f_2 = 0, \\ 4d_3c_3 - 15a_2e_1 + 15a_1e_2 - 15a_1f_1 - 15a_2f_2 = 0, \\ e_1^2 + e_2^2 - f_1^2 - f_2^2 = 0, \\ e_1f_1 + e_2f_2 = 0. \end{cases} \tag{2}$$

Let $f_1 = -e_2$, $f_2 = e_1$. Then the system (2) is changed into

$$\begin{cases} 16c_3^2 - d_3^2 + 60a_1e_1 + 60a_2e_2 = 0, \\ 2d_3c_3 - 15a_2e_1 + 15a_1e_2 = 0. \end{cases}$$

Hence,

$$c_3 = \frac{\sqrt{30}}{4}\sqrt{\sqrt{(a_1^2 + a_2^2)(e_1^2 + e_2^2)} - (a_1e_1 + a_2e_2)},$$
$$d_3 = -\sqrt{30}\sqrt{\sqrt{(a_1^2 + a_2^2)(e_1^2 + e_2^2)} + (a_1e_1 + a_2e_2)}.$$



a) $a_1 = e_1 = 1$ b) $a_1 = 1, e_1 = -1$

Fig. 1. Two examples of $\boldsymbol{r}_1(u, v)$. Here $u, v \in [-4, 4]$.

Then we obtain a class of minimal surface with four shape parameters a_1, a_2, e_1 and e_2:

$$\boldsymbol{r}(u, v) = \big(X(u, v), Y(u, v), Z(u, v)\big), \tag{3}$$

where

$$\begin{aligned} X(u, v) &= a_1\big(u^5 - 10u^3v^2 + 5uv^4\big) - a_2\big(v^5 - 10v^3u^2 + 5vu^4\big) + e_1u\big(u^2 - 3v^2\big) - e_2v\big(v^2 - 3u^2\big), \\ Y(u, v) &= a_2\big(u^5 - 10u^3v^2 + 5uv^4\big) + a_1\big(v^5 - 10v^3u^2 + 5vu^4\big) + e_2u\big(u^2 - 3v^2\big) + e_1v\big(v^2 - 3u^2\big), \\ Z(u, v) &= \frac{\sqrt{30}}{4}\sqrt{\sqrt{\big(a_1^2 + a_2^2\big)\big(e_1^2 + e_2^2\big)} - (a_1e_1 + a_2e_2)}\big(u^4 - 6u^2v^2 + v^4\big) \\ &\quad - \sqrt{30}\sqrt{\sqrt{\big(a_1^2 + a_2^2\big)\big(e_1^2 + e_2^2\big)} + (a_1e_1 + a_2e_2)}uv\big(u^2 - v^2\big). \end{aligned}$$

When $a_2 = e_2 = 0$, $\boldsymbol{r}(u, v)$ in (3) is changed into

$$\boldsymbol{r}_1(u, v) = \big(X_1(u, v), Y_1(u, v), Z_1(u, v)\big), \tag{4}$$

where

$$\begin{aligned} X_1(u, v) &= a_1\big(u^5 - 10u^3v^2 + 5uv^4\big) + e_1u\big(u^2 - 3v^2\big), \\ Y_1(u, v) &= a_1\big(v^5 - 10v^3u^2 + 5vu^4\big) + e_1v\big(v^2 - 3u^2\big), \\ Z_1(u, v) &= \frac{\sqrt{30}}{4}\sqrt{\sqrt{a_1^2e_1^2} - a_1e_1}\big(u^4 - 6u^2v^2 + v^4\big) - \sqrt{30}\sqrt{\sqrt{a_1^2e_1^2} + a_1e_1}uv\big(u^2 - v^2\big). \end{aligned}$$

When $a_1e_1 > 0$ or $a_1e_1 < 0$, $\boldsymbol{r}_1(u, v)$ has two different forms. In the cases $a_1e_1 > 0$, the minimal surface $\boldsymbol{r}_1(u, v)$ is denoted by $\overline{\boldsymbol{r}_1}(u, v) = (\overline{X_1}(u, v), \overline{Y_1}(u, v), \overline{Z_1}(u, v))$. The Gaussian curvature of $\overline{\boldsymbol{r}_1}(u, v)$ is

$$\overline{K_1} = -60a_1e_1\big(u^2 + v^2\big)^2. \tag{5}$$

Fig. 1a shows an example of $\overline{\boldsymbol{r}_1}(u, v)$ with $a_1 = e_1 = 1$.

When $a_1e_1 < 0$, we denote $\boldsymbol{r}_1(u, v)$ by $\underline{\boldsymbol{r}_1}(u, v) = (\underline{X_1}(u, v), \underline{Y_1}(u, v), \underline{Z_1}(u, v))$. Fig. 1b presents an example of $\underline{\boldsymbol{r}_1}(u, v)$ with $a_1 = 1, e_1 = -1$.

Enneper surface has several interesting properties, such as symmetry, self-intersection on symmetric planes, and containing straight lines. For $\overline{\boldsymbol{r}_1}(u, v)$ and $\underline{\boldsymbol{r}_1}(u, v)$, they have the similar properties.

Proposition 1. *The minimal surface $\underline{\boldsymbol{r}_1}(u, v)$ is symmetric about the plane $X = 0$, the plane $Y = 0$, the plane $X = Y$ and the plane $X = -Y$.*

Fig. 2a shows the minimal surface and its symmetric planes.

Proposition 2. *$\underline{\boldsymbol{r}_1}(u, v)$ is a kind of minimal surface with self-intersections, and the self-intersection points are only on the symmetric planes, i.e., there are no other self-intersection points on $\underline{\boldsymbol{r}_1}(u, v)$.*

Proof. Suppose there is a self-intersection point $\boldsymbol{p}$, which is not on the symmetric planes of $\underline{\boldsymbol{r}_1}(u, v)$. From the parametric form of $\underline{\boldsymbol{r}_1}(u, v)$, we can find a plane $\boldsymbol{L}$ such that $\boldsymbol{p}$ is on $\boldsymbol{L}$ and L is the symmetric plane of $\underline{\boldsymbol{r}_1}(u, v)$. This is contrary to Proposition 1. Hence, the self-intersection points are only on the symmetric planes. This completes the proof. □

Figs. 2b–2d illustrate the self-intersection curves on the minimal surfaces. The self-intersection curve has the same symmetric plane as the minimal surface.

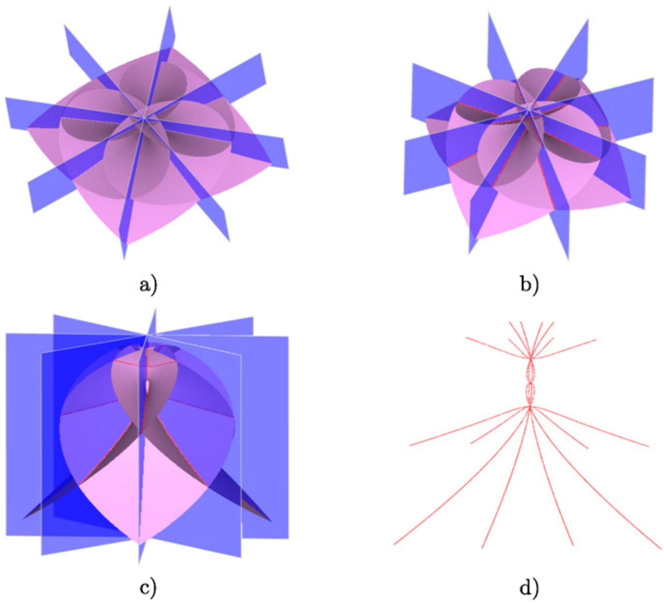


Fig. 2. Symmetric planes and self-intersection curves: (a) $\underline{\boldsymbol{r}_1}(u, v)$ with $a_1 = 1$ and $e_1 = -10$, $u, v \in [-4, 4]$ and its symmetric planes; (b) $\underline{\boldsymbol{r}_1}(u, v)$ with symmetric planes and self-intersection curves (red); (c) another view; (d) self-intersection curves. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

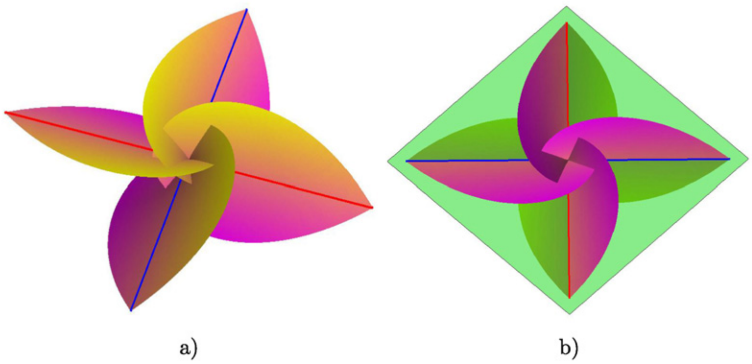


Fig. 3. The minimal surface $\overline{\boldsymbol{r}_1}(u, v)$ and the straight lines on it: (a) $\overline{\boldsymbol{r}_1}(u, v)$ with $a_1 = 1$ and $e_1 = 4$, $u, v \in [-4, 4]$ and the two orthogonal straight lines; (b) another view with plane $Z = 0$.

Proposition 3. *The minimal surface $\overline{\boldsymbol{r}_1}(u, v)$ contains two orthogonal straight lines $x = \pm y$ on the plane $Z = 0$.*

Proof. Supposing $u = \pm v$ in $\overline{\boldsymbol{r}_1}(u, v)$, we have $\overline{X_1}(u, v) = \pm\overline{Y_1}(u, v)$, $\overline{Z_1}(u, v) = 0$. Obviously, they are two orthogonal straight lines $x = \pm y$ on the plane $Z = 0$. $\square$

Fig. 3 shows the minimal surface and the straight lines on it. It is consistent with the fact that if a piece of a minimal surface has a straight line segment on its boundary, then 180° rotation around this segment is the analytic continuation of the surface across this edge.

When $a_1 = e_1 = 0$, we denote the minimal surface $\boldsymbol{r}(u, v)$ in (3) by $\boldsymbol{r}_2(u, v)$. It has the similar properties with $\boldsymbol{r}_1(u, v)$ as presented in Propositions 1, 2 and 3.

In the case $a_2 e_2 < 0$, the minimal surface $\boldsymbol{r}_2(u, v)$ is denoted by $\underline{\boldsymbol{r}_2}(u, v) = (\underline{X_2}(u, v), \underline{Y_2}(u, v), \underline{Z_2}(u, v))$. The Gaussian curvature of $\underline{\boldsymbol{r}_2}(u, v)$ is

$$\underline{K_2} = 60 a_2 e_2 \left(u^2 + v^2\right)^2. \tag{6}$$

Helicoid and catenoid are a pair of conjugate minimal surfaces. For $\boldsymbol{r}(u, v)$, we can find out a new pair of conjugate minimal surfaces as follows.

Proposition 4. *When $a_2 = -a_1$, $e_2 = e_1$, $\overline{\boldsymbol{r}_1}(u, v)$ and $\underline{\boldsymbol{r}_2}(u, v)$ are conjugate minimal surfaces.*

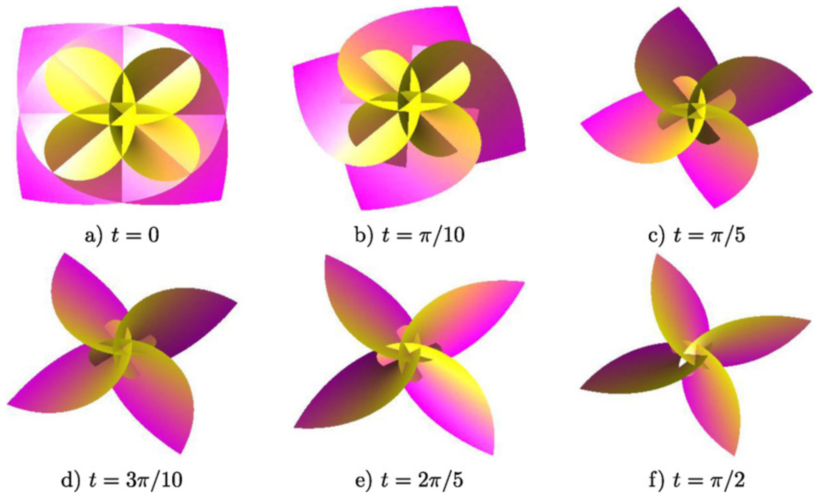


Fig. 4. Dynamic deformation between $\overline{\boldsymbol{r}_1}(u, v)$ and $\underline{\boldsymbol{r}_2}(u, v)$. Here $u, v \in [-4, 4]$.

Proof. After some computation, we have

$$\begin{aligned}
\frac{\partial \overline{X_1}(u, v)}{\partial u} &= 5a_1\left(u^4 - 6u^2v^2 + v^4\right) + 3e_1\left(u^2 - v^2\right), \\
\frac{\partial \overline{X_1}(u, v)}{\partial v} &= 20a_1uv\left(u^2 - v^2\right) - 6e_1uv, \\
\frac{\partial \underline{X_2}(u, v)}{\partial u} &= -20a_2uv\left(u^2 - v^2\right) + 6e_2uv, \\
\frac{\partial \underline{X_2}(u, v)}{\partial v} &= -5a_2\left(u^4 - 6u^2v^2 + v^4\right) + 3e_2\left(u^2 - v^2\right).
\end{aligned}$$

When $a_2 = -a_1$, $e_2 = e_1$, $\frac{\partial \overline{X_1}(u,v)}{\partial u} = \frac{\partial \underline{X_2}(u,v)}{\partial v}$, $\frac{\partial \overline{X_1}(u,v)}{\partial v} = -\frac{\partial \underline{X_2}(u,v)}{\partial u}$. That is, $\overline{X_1}(u, v)$ and $\underline{X_2}(u, v)$ are harmonic conjugate. Similarly, $\overline{Y_1}(u, v)$ and $\underline{Y_2}(u, v)$, $\overline{Z_1}(u, v)$ and $\underline{Z_2}(u, v)$ are also harmonic conjugate respectively. From Definition 5, the proof is completed. □

From Lemma 2, when $a_2 = -a_1$, $e_2 = e_1$, the surfaces of one-parametric family

$$\boldsymbol{r}_t(u, v) = (\cos t)\overline{\boldsymbol{r}_1}(u, v) + (\sin t)\underline{\boldsymbol{r}_2}(u, v)$$

are minimal surfaces with the same first fundamental form. These minimal surfaces are isometric and have the same Gaussian curvature at corresponding points. It is consistent with (5) and (6).

Let $t \in [0, \pi/2]$. When $a_2 = -a_1$ and $e_1 = e_2$, for $t = 0$, the minimal surface $\boldsymbol{r}_t(u, v)$ reduces to $\overline{\boldsymbol{r}_1}(u, v)$; for $t = \pi/2$, it reduces to $\underline{\boldsymbol{r}_2}(u, v)$. Then when t varies from 0 to $\pi/2$, $\overline{\boldsymbol{r}_1}(u, v)$ can be continuously deformed into $\underline{\boldsymbol{r}_2}(u, v)$, and each intermediate surface is also minimal surface. Fig. 4 illustrates the isometric deformation when $a_1 = -a_2 = 1$, $e_1 = e_2 = -10$. It is similar with the isometric deformation between helicoid and catenoid [14].

We can also derive some other new minimal surfaces from (1) (see Appendix A). They have four shape parameters, and also have the similar properties with $\boldsymbol{r}(u, v)$.

5. Conclusion and future work

In this paper, quintic parametric polynomial minimal surface is studied. We firstly propose the sufficient condition of a quintic harmonic polynomial parametric surface being a minimal surface. Then new minimal surfaces with several shape parameters are obtained from this condition. We analyze the properties of the new minimal surfaces, such as symmetry, self-intersection on symmetric planes and containing straights lines. In particular, we also construct two one-parameter families of isometric minimal surfaces, and implement the isometric deformation between them.

As a part of our future work, we will study the parametric polynomial minimal surface of general degree and find a unified parametric form, which can be used directly in CAD systems.

Acknowledgements

The authors would like to thank Professor Bernard Mourrain and Professor Wenping Wang for their valuable comments and suggestions. The work described in this article is supported by the National Nature Science Foundation of China (Nos. 60970079, 60773179, 60473130), the Foundation of State Key Basic Research 973 Development Programming Item of China (No. G2004CB318000), and the Zhejiang Provincial Natural Science Foundation of China under Grant No. Y1090718.

Appendix A

A class of quintic minimal surface with four shape parameters a_1, a_2, i_1 and i_2:

$$\boldsymbol{R}(u,v) = \big(X_R(u,v), Y_R(u,v), Z_R(u,v)\big),$$

where

$$\begin{aligned}
X_R(u,v) &= a_1\big(u^5 - 10u^3v^2 + 5uv^4\big) - a_2\big(v^5 - 10v^3u^2 + 5vu^4\big) + i_1u + i_2v,\\
Y_R(u,v) &= a_2\big(u^5 - 10u^3v^2 + 5uv^4\big) + a_1\big(v^5 - 10v^3u^2 + 5vu^4\big) + i_2u - i_1v,\\
Z_R(u,v) &= \frac{\sqrt{10}}{3}\sqrt{\sqrt{\big(a_1^2+a_2^2\big)\big(i_1^2+i_2^2\big)} - (a_1i_1 + a_2i_2)}\,u\big(u^2 - 3v^2\big)\\
&\quad - \frac{\sqrt{10}}{3}\sqrt{\sqrt{\big(a_1^2+a_2^2\big)\big(i_1^2+i_2^2\big)} + (a_1i_1 + a_2i_2)}\,v\big(v^2 - 3u^2\big).
\end{aligned}$$

References

[1] K.W. Bletzinger, Form finding of membrane structures and minimal surfaces by numerical continuation, in: Proceeding of the IASS Congress on Structural Morphology: Towards the New Millennium, Nottingham, ISBN 0-85358-064-2, 1997, pp. 68–75.
[2] W.Y. Chen, Y.Y. Cai, J.M. Zheng, Constructing triangular meshes of minimal area, Computer-Aided Design and Applications 5 (14) (2008) 508–518.
[3] M. Desbrun, E. Grinspun, P. Schroder, Discrete Differential Geometry: An Applied Introduction, SIGGRAPH Course Notes, 2005.
[4] M. Do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, 1976.
[5] G. Farin, Curves and Surfaces for CAGD, 5th ed., Morgan Kaufmann, 2002.
[6] S. Grandine, T. Del Valle, S.K. Moeller, G. Natarajan, J. Pencheva, S. Sherman Wise, Designing airplane struts using minimal surfaces, IMA Preprint 1866, 2002.
[7] X.F. Gu, S.T. Yau, Surface classification using conformal structures, in: ICCV, 2003, pp. 701–708.
[8] X.F. Gu, S.T. Yau, Computing conformal structure of surfaces, CoRR, cs.GR/0212043, 2002.
[9] X.F. Gu, Y. He, H. Qin, Manifold splines, Graphical Models 68 (3) (2006) 237–254.
[10] M. Jin, F. Luo, X.F. Gu, Computing general geometric structures on surfaces using Ricci flow, Computer-Aided Design 8 (2007) 663–675.
[11] K. Jung, K.T. Chu, S. Torquato, A variational level set approach for surface area minimization of triply-periodic surfaces, Journal of Computational Physics 2 (2007) 711–730.
[12] X. Li, X.H. Guo, H.Y. Wang, Y. He, X.F. Gu, H. Qin, Harmonic volumetric mapping for solid modeling applications, in: Symposium on Solid and Physical Modeling, 2007, pp. 109–120.
[13] J.J. Man, G.Z. Wang, Polynomial minimal surface in isothermal parameter, Journal of Computers 25 (2) (2002) 197–201.
[14] J.C.C. Nitsche, Lectures on Minimal Surfaces, vol. 1, Cambridge Univ. Press, Cambridge, 1989.
[15] R. Osserman, A Survey of Minimal Surfaces, 2nd ed., Dover Publ., New York, 1986.
[16] U. Pinkall, K. Polthier, Computing discrete minimal surface and their conjugates, Experimental Mathematics 1 (1993) 15–36.
[17] G. Slabaugh, B. Whited, J. Rossignac, T. Fang, G. Unal, 3D ball skinning using PDEs for generation of smooth tubular surfaces, Computer-Aided Design 1 (2010) 18–26.
[18] J. Sullivan, The aesthetic value of optimal geometry, in: M. Emmer (Ed.), The Visual Mind II, MIT Press, 2005, pp. 547–563.
[19] J. Wallner, H. Pottmann, Infinitesimally flexible meshes and discrete minimal surfaces, Monatsh. Math. 153 (2008) 347–365.
[20] Y. Wang, Periodic surface modeling for computer aided nano design, Computer-Aided Design 3 (2007) 179–189.
[21] G.L. Xu, Q. Zhang, G^2 surface modeling using minimal mean-curvature-variation flow, Computer-Aided Design 5 (2007) 342–351.
[22] G.L. Xu, Geometric Partial Differential Equation Methods in Computational Geometry, Science Press, Beijing, 2008.
[23] G.L. Xu, Discrete Laplace-Beltrami operators and their convergence, Computer Aided Geometric Design 10 (2004) 767–784.
[24] G.L. Xu, Convergence analysis of a discretization scheme for Gaussian curvature over triangular surfaces, Computer Aided Geometric Design 2 (2006) 193–207.